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A Boundary Value Problem and Expansion Formula of I - Function and General Class of Polynomials with Application

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Abstract
In the present paper, we make a model of a boundary value problem and then obtain its solution involving products of I -function and a general class of polynomials.
Key words: General Class of Polynomials, I -function, Expansion Formula, Hermite Polynomial.
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I. Introduction

The operational techniques are important tools to compute various problems in various fields of sciences which are used in the works of Chaurasia [4], Chandel, Agrawal and Kumar [2], Chandel and Sengar [3] and Kumar [5] to find out several results in various problems in different field of sciences and thus motivating by this work, we construct a model problem for temperature distribution in a rectangular plate under prescribed boundary conditions and then evaluate its solution involving A-function with product of general class of polynomials.

The general class of polynomials is defined by Srivastava and Panda [11, 12] as:

$$S_{n_1,\dots,n_r}^{m_1,\dots,m_r}(x_1,\dots,x_r)=\sum_{k_1=0}^{[n_1/m_1]}\dots\sum_{k_r=0}^{[n_r/m_r]}\frac{(-n_1)_{m_1k_1}}{k_1!}\dots\frac{(-n_r)_{m_rk_r}}{k_r!}F[n_1,k_1;\dots;n_r,k_r]x_1^{k_1}\dots x_r^{k_r}\quad , \tag{1.1}$$

Where, $m_1,\dots,m_r$ are arbitrary positive integers and the coefficients $F[n_1,k_1;\dots;n_r,k_r]$ are arbitrary constants real or complex . Finally, we derive some new particular cases and find their applications also.
The I-function introduced by Saxena [6] will be represented and defined as follows :

$$I[Z]=I_{p_i,q_i:r}^{m,n}[Z]=I_{p_i,q_i:r}^{m,n}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]=\frac{1}{2\pi\omega}\int_L\chi(\xi)d\xi \tag{1.2}$$

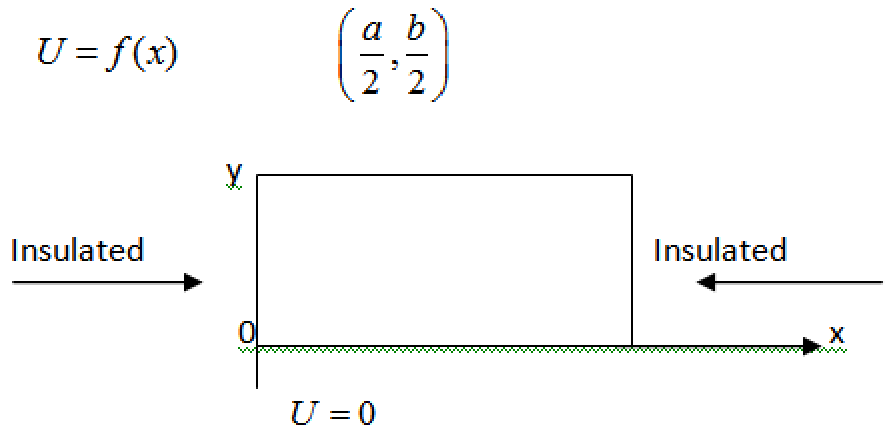
where $\omega=\sqrt{-1}$

$$\chi(\xi)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-\beta_{ji})\prod_{j=n+1}^{p_i}\Gamma(a_{ji},\alpha_{ji})\right\}} \tag{1.3}$$

$p_i,q_i(i=1,\dots,r),m,n$ are integers satisfying $0\le n\le p_i$, $0\le m\le q_i,(i=1,\dots,r),r$ is finite $\alpha_j,\beta_j,\alpha_{ij},\beta_{ji}$ are real and a_j,b_j,a_{ji},b_{ji} are complex numbers such that
$\alpha_j(b_h+v)\ne\beta_h(a_j-v-k)$ for $v,k=01,2,\dots$
We shall use the following notations:
$A^*=(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i};B^*=(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}$

II. A Boundary Value Problem

We consider a rectangular plate such that,



Where the boundary value conditions are:

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=a,\ 0<x<\frac{a}{2}, 0<y<\frac{b}{2} \tag{2.1}$$

$$\left.\frac{\partial U}{\partial x}\right|_{x=0}=\left.\frac{\partial U}{\partial x}\right|_{x=\frac{a}{2}}=0, 0<y<\frac{b}{2} \tag{2.2}$$

$$U(x,0)=0, 0<x<\frac{a}{2} \tag{2.3}$$

$$U\left(x,\frac{b}{2}\right)=f(x)=\left(\cos\frac{\pi x}{a}\right)S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}\left[y_1\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right] A_{p,q}^{m,n}\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\left|\begin{matrix}{}_1(a_j,\alpha_j)_p\\ {}_1(b_j,\beta_j)_q\end{matrix}\right.\right] \tag{2.4}$$

Where, $0<x<\frac{a}{2}$ provided that $\mathrm{Re}(\eta)>-1, \sigma>0.$

$U(x,y)$ is the temperature distribution in the rectangular plate at point (x,y).

III. Main Integral

In our investigations, we make an appeal to the modified formula due to Kumar [5] as,

$$\int_0^{a/2}\left(\cos\frac{\pi x}{a}\right)^{\eta}\cos\frac{2m\pi x}{a}\,dx=\frac{a\Gamma(\eta+1)}{2^{\eta+1}\left(\frac{\eta}{2}+m+1\right)\left(\frac{\eta}{2}-m+1\right)} \tag{3.1}$$

Where, m is positive integer and $\mathrm{Re}(\eta)>-1$, then we evaluate an applicable integral

$$\int_0^{a/2}\left(\cos\frac{\pi x}{a}\right)^{\eta}\cos\frac{2m\pi x}{a}S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}\left[y\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right] I_{p_i,q_i:r}^{m,n}\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\left|\begin{matrix}A*\\ B*\end{matrix}\right.\right]dx$$

$$= \frac{a}{2^{\eta+1}} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_r=0}^{[n_r/m_r]} \frac{(-n)_{m_1k_1}}{k_1!} \cdots \frac{(-n)_{m_rk_r}}{k_r!}$$

$$F[n_1,k_1;...,n_r,k]I(k)\left(\frac{y}{4\rho}\right)^{k_1}\cdots\left(\frac{y}{4\rho}\right)^{k_r} \tag{3.2}$$

Where,

$$I(k) = I^{m,n+1}_{p_i+1,q_i+1:r}\left[\frac{z}{4^{\sigma}}\left|\begin{matrix}(-\eta-2\rho k_1-...-2\rho k_r;2\sigma), A^* \\ B^*,\left(-\frac{\eta}{2}-m-\rho k_1-...-\rho k_r;\sigma\right)\end{matrix}\right.\right] \tag{3.3}$$

Provided that $F[n_1,k_1;...;n_r,k_r]$ are arbitrary functions of $n_1,k_1;...;n_r,k_r$, real or complex independent of x, y, ρ, the conditions of (2.4) and (3.1) are satisfied and

$$\mathrm{Re}\left(\eta+\sigma\frac{b_{ji}}{\beta_{ji}}\right) > -1, \; |\arg z| \le \frac{1}{2}\pi\Omega,$$

Where

$$\Omega \equiv \sum_{j=1}^{n}\alpha_j + \sum_{j=1}^{m}\beta_j - \sum_{j=n+1}^{p_i}\alpha_{ji} - \sum_{j=m+1}^{q_i}\beta_{ji} > 0$$

IV. Solution of Boundary Value Problem

In this section, we obtain the solution of the boundary value problem stated in the section (2) as using (2.1), (2.2) and (2.3) with the help of the techniques referred to Zill [13] as:

$$U(x,y) = A_0 y + \sum_{p=1}^{\infty} A_p \sinh\frac{2p\pi y}{a}\cos\frac{2p\pi x}{a}, 0<x<\frac{a}{2}, 0<y<\frac{a}{2} \tag{4.1}$$

For $y=\frac{b}{2}$, we find that

$$U\left(x,\frac{b}{2}\right) = f(x) = \frac{A_0 b}{2} + \sum_{p=1}^{\infty} A_p \sinh\frac{p\pi b}{a}\cos\frac{2p\pi x}{a}, 0<x<\frac{a}{2} \tag{4.2}$$

Now making an appeal to (2.4) and (4.2) and then interchanging both sides with respect to x from 0 to $\frac{a}{2}$, we derive,

$$A_0 = \frac{2}{b\sqrt{\pi}} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_r=0}^{[n_r/m_r]} (-n_1)_{m_1k_1}\cdots(-n_r)_{m_rk_r}$$

$$F[n_1,k_1;...;n_r,k_r]I_1(k)\frac{y^{k_1}}{k_1!}\cdots\frac{y^{k_r}}{k_r!} \tag{4.3}$$

Where

$$I_1(k) = I^{m,n+1}_{p_i+1,q_i+1:r}\left[z\left|\begin{matrix}\left(-\frac{1}{2}-\frac{\eta}{2}-\rho k_1-...-\rho k_r;\sigma\right), A^* \\ B^*,\left(-\frac{\eta}{2}-\rho k_1-...-\rho k_r;\sigma\right)\end{matrix}\right.\right] \tag{4.4}$$

Where all conditions of (2.4), (3.1) and (3.3) are satisfied.

Again making an appeal to (2.4) and (4.2) and then multiplying by $\cos\frac{2m\pi x}{a}$ both sides and thus integrating that result with respect to x from 0 to $\frac{a}{2}$, we find,

$$A_m = \frac{1}{2^{\eta-1}\sinh\frac{p\pi b}{a}} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_r=0}^{[n_r/m_r]} \frac{(-n_1)_{m_1k_1}}{k_1!} \cdots \frac{(-n_r)_{m_rk_r}}{k_r!} F[n_1,k_1;\ldots,n_r,k_r]$$

$$I(k)\left(\frac{y}{4^\rho}\right)^{k_1} \cdots \left(\frac{y}{4^\rho}\right)^{k_r} \qquad (4.5)$$

Provided that all conditions of (2.4), (3.1) and (3.3) are satisfied.

Finally, making an appeal to the result (4.1), (4.3) and (4.5), we derive the required solution of the boundary value problem,

$$U(x,y) = \frac{2y}{b\sqrt{\pi}} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_r=0}^{[n_r/m_r]} \left[\prod_{j=1}^{r}\left((-n_j)_{m_jk_j}\frac{y^{k_j}}{k_j!}\right)\right] F[n_1,k_1;\ldots;n_r,k_r] +$$

$$\sum_{m=1}^{\infty} \frac{\sinh\frac{2m\pi y}{a}\cos\frac{2m\pi x}{a}}{2^{\eta-1}\sinh\frac{m\pi b}{a}} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_r=0}^{[n_r/m_r]} \left[\prod_{j=1}^{r}\left((-n_j)_{m_jk_j}\left(\frac{y}{4^\rho}\right)^{k_j}\frac{1}{k_j!}\right)\right]$$

$$F[n_1,k_1;\ldots;n_r,k_r]I(k) \qquad (4.6)$$

Where ,

Provided that all conditions of (2.4), (3.1) and (3.3) are satisfied.

V. Expansion Formula

With the aid of (2.4) and (4.6) and then setting $y = \frac{b}{2}$, we evaluate the expansion formula

$$\left(\cos\frac{\pi x}{a}\right)^{\eta} S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}\left[y\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]$$

$$I_{p_i,q_i:r}^{m,n}\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\middle|\begin{matrix}A*\\B*\end{matrix}\right]$$

$$= \frac{1}{\sqrt{\pi}} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_r=0}^{[n_r/m_r]} \left[\prod_{j=1}^{r}\left((-n_j)_{m_jk_j}\frac{y^{k_j}}{k_j!}\right)\right] F[n_1,k_1;\ldots;n_r,k_r]I(k)$$

$$\sum_{m=1}^{\infty} \frac{\cos\frac{2m\pi x}{a}}{2^{\eta-1}} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_r=0}^{[n_r/m_r]} \left[\prod_{j=1}^{r}\left((-n_j)_{m_jk_j}\left(\frac{y}{4^\rho}\right)^{k_j}\frac{1}{k_j!}\right)\right]$$

$$F[n_1,k_1;\ldots;n_r,k_r]I(k) \qquad (5.1)$$

where $0 < x < \frac{a}{2}$,

provided that all conditions of (2.4), (3.1) and (3.3) are satisfied.

VI. Particular Cases and Applications

In this section, we do some setting of different parameters of our results and then drive some particular cases as stated here as taking $m_1 = ... = m_r = \gamma$ and

$$F[n_1, k_1; ...; n_r, k_r] = \left(\frac{h}{(v)^\gamma}\right)^{k_1+...+k_r} \frac{1}{(1+p-n_1-...-n_r)_{\gamma(k_1+...+k_r)}}$$ in (1.1)

We get,

$$S_{n_1,...,n_r}^{\gamma,...,\gamma}[x_1,...,x_r] = \frac{(-v)^{-n_1...-n_r}}{(-p)_{n_1+...+n_r}}(x_1)^{n_1/\gamma}...(x_r)^{n_r/\gamma} H_{n_1,...,n_r}^{(h,\gamma,v,p)}[(x_1)^{-1/\gamma}...(x_r)^{-1/\gamma}] \qquad (6.1)$$

And thus, we obtain an integral for product of a class of polynomials of several variables and cosine functions as

$$\int_0^{a/2}\left(\cos\frac{\pi x}{a}\right)^\eta \cos\frac{2m\pi x}{a}\frac{(-v)^{-n_1-...-n_r}}{(-p)_{n_1+...,+n_r}}\left[y\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]^{n_1/\gamma}...\left[y\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]^{n_1/\gamma}$$

$$H_{n_1,...n_r}^{(h,\gamma,v,p)}\left[\left[y\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]^{r/\gamma}\right] I_{p_i,q_i:r}^{m,n}\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\middle|\begin{matrix}A^*\\B^*\end{matrix}\right]dx$$

$$= \frac{a}{2^{\eta+1}}\sum_{k_1=0}^{[n_1/\gamma]}...\sum_{k_r=0}^{[n_r/\gamma]}\frac{(-n)_{\gamma k_1}}{k_1!}...\frac{(-n)_{\gamma k_r}}{k_r!}\left[\frac{h}{(-v)^\gamma}\right]^{k_1+...+k_r}$$

$$\frac{1}{(1+\rho-n_1-...-n_r)_{\gamma(k_1+...+k_r)}} I(k)\left(\frac{y}{4\rho}\right)^{k_1}...\left(\frac{y}{4\rho}\right)^{k_r} \qquad (6.2)$$

Provided that all conditions of (2.4), (3.1) and (3.2) are satisfied.
The solution of the given problem is

$$U(x,y) = \frac{2y}{b\sqrt{\pi}}\sum_{k_1=0}^{[n_1/\gamma]}...\sum_{k_r=0}^{[n_r/\gamma]}\left[\prod_{j=1}^{r}\left((-n_j)_{\gamma k_j}\left(\frac{hy}{(-v)^\gamma}\right)^{k_j}\frac{1}{k_j!}\right)\right]$$

$$\frac{1}{(1+p-n_1-...-n_r)_{\gamma(k_1+...+k_r)}} I_1(k) + \sum_{m=1}^{\infty}\frac{\sinh\frac{2m\pi y}{a}\cos\frac{2m\pi x}{a}}{2^{\eta-1}\sinh\frac{m\pi b}{a}}$$

$$\frac{1}{(1+p-n_1-...n_r)_{\gamma(k_1+...+k_r)}} I(k) \qquad (6.3)$$

When $0 < x < \frac{a}{2}, 0 < y < \frac{b}{2}$, provided that all conditions of (2.4), (3.1) and (3.3) are satisfied.

The expansion formula is

$$\left(\cos\frac{\pi x}{a}\right)^\eta \frac{(-v)^{-n_1-...-n_r}}{(-p)_{n_1+...,+n_r}}\left[y\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]^{n_1/\gamma}...\left[y\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]^{n_1/\gamma}$$

$$H_{n_1,...n_r}^{(h,\gamma,v,p)}\left[\left[y\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]^{-r/\gamma}\right] I_{p_i,q_i:r}^{m,n}\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\middle|\begin{matrix}A^*\\B^*\end{matrix}\right]$$

$$= \frac{1}{\sqrt{\pi}}\sum_{k_1=0}^{[n_1/\gamma]}...\sum_{k_r=0}^{[n_r/\gamma]}\left[\prod_{j=1}^{r}\left((-n_j)_{\gamma k_j}\left(\frac{hy}{(-v)^\gamma}\right)^{k_j}\frac{1}{k_j!}\right)\right]\frac{1}{(1+p-n_1-...-n_r)_{\gamma(k_1+...+k_r)}}$$

$$I_1(k)+\sum_{m=1}^{\infty}\frac{\cos\frac{2m\pi x}{a}}{2^{\eta-1}}\sum_{k_1=0}^{[n_1/\gamma]}\cdots\sum_{k_r=0}^{[n_r/\gamma]}\left[\prod_{j=1}^{r}\left((-n_j)_{\gamma k_j}\left(\frac{hy}{(-v)^{\gamma}4^{\rho}}\right)^{k_j}\frac{1}{k_j!}\right)\right]$$
$$\frac{1}{(1+p-n_1-...n_r)_{\gamma(k_1+...+k_r)}}I(k) \tag{6.4}$$

When $0<x<\frac{a}{2}$, provided that all conditions of (2.4), (3.1) and (3.3) are satisfied. Further, making an use of the result due to Chandel, Agarwal and Kumar ([1]p. 27, wq. (1.4) and (1.5)).

$$\lim_{p\to\infty}H_{n_1,...,n_r}^{(h,\gamma,1,p)}\left(\frac{x_1}{p},...,\frac{x_r}{p}\right)=,$$
$$\lim_{p\to\infty}H_{n_1,...,n_r}^{(h,\gamma,1/p,p)}\left(x_1,...,x_r\right)=g_{n_1}^{\gamma}(x_1,h)...g_{n_r}^{\gamma}(x_r,h) \tag{6.5}$$

To the results (6.2), (6.3) and (6.4), we get another different relation in similar way.
Further again, applying the relation

$$g_n^2(x,-1/4)=2^{-n}H_n(x) \tag{6.6}$$

To the above results, we evaluate another result for Hermite polynomials by same techniques.
Other special cases and applications of our results may be obtained by making use of the work of Chandel and Sengar [3], Srivastava and Karlsson [9] and Srivastava and Manocha [10] , due to lack of space we omit them.

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